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Deformations of the Hodge map and optical geometry

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Abstract. *A simple formula is derived for the infinitesimal change of the Hodge dual of a k-form, induced by a deformation of the scalar product in the underlying vector space. By considering deformations due to a flow generated by a vector field on a differential manifold, one obtains an expression for the commutator of the Hodge dual with the Lie derivation with respect to the vector field, acting on differential forms. This formula is useful in proving theorems on optical solutions of Maxwell's and Yang-Mills equations. The optical geometry underlying such solutions is defined as a restriction of the bundle of linear frames of a 4-dimensional manifold to a 9-dimensional optical group. This geometry provides a natural framework for the study of shearfree, optical and geodesic congruences and of the associated fields.*

1. INTRODUCTION

In theoretical physics one often considers mathematical models of the following type. There is given an n-dimensional smooth manifold M, a Lie group G, a principal G-bundle $P \to M$ and a representation of G in a finite-dimensional, real or complex vector space V. Physical histories (classical fields, wave-functions, etc.) are described by V-valued k-forms on P, equivariant under the action of G. For example, a connection on P is described by a $\mathfrak{g}$-valued 1-form ω which corresponds

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to the adjoint representation of G in its Lie algebra $\mathfrak{g}$ and, moreover, is a left inverse for the map $\mathfrak{g} \to TP$ defined by the action of G in P. Of special interest are horizontal k-forms; the curvature 2-form

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]$$

is horizontal and, if ϕ is a horizontal k-form, then its covariant exterior derivative $D\phi$ is a horizontal $(k+1)$-form. If M is oriented and has a (Riemannian or Lorentzian) metric tensor g, then the Hodge dual map $*$ can be applied to horizontal forms on P: if ϕ is a horizontal V-valued k-form on P, then $*\phi$ is a similar $(n-k)$-form. Many fundamental equations of physics have the following structure

$$(1) \qquad D * \phi = * j,$$

where ϕ and j are horizontal, V-valued k- and $(k-1)$-forms, respectively. For example, if $G = U(1)$, ϕ is the curvature 2-form and j is the $\mathbb{R}$-valued 1-form of electric current, then (1) is simply the Maxwell equation. For a non-Abelian group G and $\phi = \Omega$ equation (1) coincides with that introduced by Yang and Mills. If P is the bundle of linear frames of M endowed with a linear connection and $\theta = (\theta^\mu)$ denotes the soldering form, then the choice $\phi = (\theta^\mu \wedge \theta^\nu)$, $\mu, \nu = 1, \ldots, n$, leads to the Cartan equation of a relativistic theory of gravitation with spin j and torsion $D\theta$ [1 – 3].

In view of the occurrence of the Hodge map in the fundamental equation (1) it is interesting to consider the dependence of $*$ on g and, in particular, its behaviour under deformations of the metric. This results in a formula for the commutator of $*$ with Lie derivation relative to a vector field. The formula has already been used to prove a theorem on the existence of optical (isotropic) Yang-Mills configurations associated with shear-free congruences of null geodesics [4]. The paper is concluded with a section on the «optical geometry» underlying the local structure of such congruences [5 - 7].

2. THE HODGE MAP AND ITS DEFORMATIONS

Let V be an n-dimensional real vector space with a preferred orientation. The group $GL(n, \mathbb{R})$ acts transitively in the manifold $F(V)$ of all linear frames in V, similarly, $GL^+(n, \mathbb{R})$ acts transitively in the open submanifold $F^+(V) \subset F(V)$ of frames with the preferred orientation. A scalar product in V is defined as a symmetric bilinear map $g : V \times V \to \mathbb{R}$ which is nonsingular: if $g(u, v) = 0$ for all $u \in V$, then $v = 0$. Let $S(V) \subset V^* \otimes V^*$ be the set (in fact, manifold) of all scalar products in V. If $e = (e_\mu)$, $\mu = 1, \ldots, n$, is a frame and $g \in S(V)$ then the formula

$$g_{\mu\nu}(e) = g(e_\mu, e_\nu)$$

defines the functions $g_{\mu\nu} : S(V) \to \mathbb{R}$ and

$$g_{\mu\nu}(ea) = g_{\rho\sigma}(e)\, a^\rho_\mu\, a^\sigma_\nu ,$$

where $a = (a^\rho_\sigma) \in GL(n, \mathbb{R})$. If $\gamma(e) = \det(g_{\mu\nu}(e))$, then

$$\gamma(ea) = \gamma(e)\,(\det a)^2$$

and the sign of $\gamma(e)$ is an invariant.

The *Grassmann algebra* of forms over V in denoted by

$$\Lambda V^* = \bigoplus_{k=0}^{n} \Lambda^k V^*,$$

where $\Lambda^0 V^* = \mathbb{R}$ and $\Lambda^1 V^* = V^*$ is the dual of V. For any frame $(e_\mu) \in F(V)$ its dual $(e^\mu) \in F(V^*)$ is determined by

$$\langle e_\mu, e^\nu \rangle = \delta^\nu_\mu,$$

where angular brackets denote the evaluation map $V \times V^* \to \mathbb{R}$.

Let $(e_\mu) \in F^+(V)$ and (e^μ) be its dual. The volume form

$$\mathrm{vol}\,(g) = |\gamma(e)|^{1/2}\, e^1 \wedge e^2 \wedge \ldots \wedge e^n$$

depends on $g \in S(V)$, but not on the frame, provided it is of preferred orientation.

A convenient abuse of notation consists in using the same letter

(2) $$g : V \to V^*$$

for the linear map defined by

$$\langle u, g(v) \rangle = g(u, v)$$

as for the scalar product $g \in S(V)$ itself.

For any k-form α its *Hodge dual* $\sigma(g)\alpha$ is defined as the $(n-k)$-form given by its value on the vectors $u_{k+1}, \ldots, u_n \in V$ as follows:

(3) $$\mathrm{vol}\,(g) \cdot \sigma(g)\alpha\,(u_{k+1}, \ldots, u_n) = \alpha \wedge g(u_{k+1}) \wedge \ldots \wedge g(u_n).$$

When g is fixed once for all, then one usually writes $*\,\alpha$ instead of $\sigma(g)\alpha$. The latter, more elaborate notation is used here in order to study the dependence of the Hodge map on g. Clearly, $\sigma(g)$ can be extended to a linear map

$$\sigma(g) : \Lambda V^* \to \Lambda V^*$$

and it is known that

(4) $$\sigma(g)^2\alpha = (-1)^{k(n+1)} \operatorname{sgn} \gamma(e)\alpha \qquad \text{for } \alpha \in \Lambda^k V^*.$$

Let $t \to g_t (t \in \mathbb{R})$ be a smooth curve in $S(V)$. (In fact, only the first jet of $t \to g_t$ at $t = 0$ will be used). If g^{-1} denotes the inverse of (2), then the composed map $g_t \circ g^{-1}$ is an automorphism of V^*,

(5) $$h = \frac{\mathrm{d}}{\mathrm{d}t} g_t \circ g^{-1} \bigg|_{t=0}$$

is an endomorphism of V^*, and

(6) $$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{vol}(g_t) \bigg|_{t=0} = \frac{1}{2} \operatorname{Tr} h \cdot \operatorname{vol}(g_0).$$

The derivation of the Grassmann algebra ΛV^* induced by h is denoted $i(h)$. This is a linear map such that, if $\alpha \in V^*$, then $i(h)\alpha = h(\alpha)$, and

(7) $$i(h)(\phi \wedge \psi) = (i(h)\phi) \wedge \psi + \phi \wedge i(h)\psi$$

for any $\phi, \psi \in \Lambda V^*$. It follows from the definition of the volume form that

(8) $$i(h) \operatorname{vol}(g) = (\operatorname{Tr} h) \operatorname{vol}(g).$$

Let g_t be now substituted for g in (3) and both sides of the resulting equation differentiated with respect to t at 0. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{vol}(g_t) \bigg|_{t=0} \sigma(g_0)\alpha(u_{k+1}, \ldots, u_n) +$$

$$+ \operatorname{vol}(g_0) \frac{\mathrm{d}}{\mathrm{d}t} \sigma(g_t)\alpha(u_{k+1}, \ldots, u_n) =$$

$$= \alpha \wedge i(h)(g_0(u_{k+1}) \wedge \ldots \wedge g_0(u_n)).$$

Taking into account equations (6 - 8) and denoting $\sigma(g_0)$ by $*$, the last formula can be written as

(9) $$\frac{\mathrm{d}}{\mathrm{d}t} \sigma(g_t)\alpha \bigg|_{t=0} = - * i(h)\alpha + \frac{1}{2} (\operatorname{Tr} h) * \alpha.$$

Moreover, it follows by differentiation of (4), with g_t substituted for g, that the linear map defined by (9) anticommutes with $*$. This is equivalent to

(10) $$* i(h)\alpha + i(h) * \alpha = (\operatorname{Tr} h) * \alpha$$

so that (9) can be rewritten as

(11) $$\left.\frac{\mathrm{d}}{\mathrm{d}t}\,\sigma(g_t)\alpha\right|_{t=0} = (i(h) - 1/2\ \mathrm{Tr}\ h) * \alpha.$$

To a conformal deformation of $g_0 \in S(V)$, $g_t = g_0$ expt, there corresponds

$$h = i\mathrm{d}.$$

Since $i(i\mathrm{d})$ restricted to $\Lambda^k V^*$ is k times the identity and $\mathrm{Tr}\,(i\mathrm{d}) = n$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\,\sigma(g_0\,\mathrm{expt})\right|_{t=0} \alpha = \left(k - \frac{1}{2}\,n\right) * \alpha$$

and one recovers the well-known result: in an even-dimensional space, the Hodge dual of forms of middle degree is invariant under conformal changes of the scalar product.

3. DEFORMATIONS GENERATED BY FLOWS

Let M be an n-dimensional oriented paracompact differential manifold. The manifold and all relevant maps are assumed to be smooth. The bundles of linear frames and of scalar products are denoted by $F(M)$ and $S(M)$, respectively. If T_xM is the tangent space to M at x, then

$$S(M) = \bigcup_{x\in M} S(T_xM),$$

with a suitable topology and differential structure. A section g of the bundle $S(M) \to M$ is a metric tensor on M. If $x \in M$ and g is a metric tensor on M, then

$$g_x \in S(T_xM)$$

is its value at x. Only a little mental effort is required to avoid confusion between g_x and g_t occurring in the preceding section. Let

$$f : M \to M$$

be a diffeomorphism and

$$T_x f : T_xM \to T_{f(x)}M$$

denote the derived map. The pull-back of g by f is the metric tensor f^*g such that, if $u, v \in T_xM$, then

(12) $$(f^*g)_x(u, v) = g_{f(x)}(T_x f(u), T_x f(v)).$$

The bundle of k-forms on M is

$$\Lambda^k(M) = \bigcup_{x \in M} \Lambda^k(T_x^*M)$$

with a suitable topology and differential structure; the bundle $\Lambda(M)$ of Grassmann algebras is similarly defined. Its sections constitute the *Cartan algebra* of differential forms. If α is a section of $\Lambda(M) \to M$, then $\alpha_x \in \Lambda(T_x^*M)$ is its value at x. If α and β are two such sections, then $\alpha \wedge \beta$ is another section defined by

$$(\alpha \wedge \beta)_x = \alpha_x \wedge \beta_x .$$

The Hodge map of differential forms is also defined in this «pointwise» manner: if α is a section of $\Lambda^k(M)$ on a manifold with a metric tensor g, then $\sigma(g)\alpha$ is a section of $\Lambda^{n-k}(M) \to M$ given by

$$(\sigma(g)\alpha)_x = \sigma(g_x)\alpha_x .$$

The pull-back $f^*\alpha$ of a differential form α is defined similarly as in (12) and the «naturality» of the Hodge map is expressed by

$$f^*(\sigma(g)\alpha) = \sigma(f^*g) f^*\alpha \tag{13}$$

for any diffeomorphism f.

Let $(f_t)_{t \in \mathbb{R}}$ denote the flow generated by a vector field X on M. The map $t \mapsto (f_t^*g)_x$ is a curve in $S(T_xM)$, i.e. a deformation of g_x in the sense of §2. The Lie derivative of g with respect to X,

$$\mathcal{L}_X g = \frac{\mathrm{d}}{\mathrm{d}t} f_t^*g \Big|_{t=0}$$

defines an endomorphism h of the bundle $T^*M \to M$ such that

$$h_x = (\mathcal{L}_X g)_x \circ g_x^{-1} ,$$

where the notation is consistent with that in (5). If f_t is substituted for f in (13) and the resulting equation differentiated with respect to t at 0, one obtains

$$\mathcal{L}_X(\sigma(g)\alpha) = \frac{\mathrm{d}}{\mathrm{d}t} \sigma(f_t^*g)\alpha \Big|_{t=0} + \sigma(g) \mathcal{L}_X \alpha$$

so that:

$$[\mathcal{L}_X, \sigma(g)]\alpha = \frac{\mathrm{d}}{\mathrm{d}t} \sigma(f_t^*g)\alpha \Big|_{t=0} .$$

Using (11) and reverting to the traditional notation, $\sigma(g)\alpha = * \alpha$, one can write

$$(14) \qquad [\mathscr{L}_X, *]\alpha = \left(i(h) - \frac{1}{2} \operatorname{Tr} h\right) * \alpha$$

where $i(h)$ is now the derivation of the Cartan algebra defined «point-wise» in terms of $i(h_x)$.

4. OPTICAL GEOMETRY

Consider a plane electromagnetic wave described by a 2-form on an oriented Lorentz 4-manifold M. Let F and g be, respectively, the values of the 2-form and of the metric tensor at $x \in M$. They are tensors over $V = T_x M$ and there exists an optical («null», «isotropic») vector $k \in V$ and a 1-form α such that

$$F = \varkappa \wedge \alpha, \quad \text{where} \quad \varkappa = g(k) \quad \text{and} \quad \langle k, \alpha \rangle = 0.$$

The dual of F is

$$* F = \varkappa \wedge \beta$$

where $\beta \in V^*$ is orthogonal to both α and $\varkappa$, and of length equal to that of α. The 1-form $\varkappa$ is defined up to a non-zero multiplier; the 1-forms α and β are defined modulo $\varkappa$ and up to a common factor. The direction K of k, as well as $L = \ker \varkappa$, are well defined by F.

On the basis of such heuristic considerations, I define an *optical structure* in a 4-dimensional real vector space V to consist of

(A) a pair of vector subspaces («a flag»), K and L, of dimension 1 and 3, respectively, and such that

$$K \subset L \subset V;$$

(B) an orientation and a conformal scalar product in the 2-dimensional vector space L/K.

Clearly, condition (B) is equivalent to giving (B′) a complex structure in L/K, i.e. a linear map

$$J : L/K \to L/K \quad \text{such that} \quad J^2 = -id.$$

If (V, K, L, J) and (V', K', L', J') are two optical structures, then $f : V \to V'$ is an optical isomorphism if it is an isomorphism of vector spaces such that

$$f(K) = K', \quad f(L) = L' \quad \text{and} \quad J' \circ \bar{f} = \bar{f} \circ J$$

where

$$\bar{f} : L/K \to L'/K' \quad \text{is given by } \bar{f}(1 \bmod K) = f(1) \bmod K', \ 1 \in L.$$

The standard optical structure on $V_0 = \mathbb{R}^4$ is given by

$$K_0 = \{x \in \mathbb{R}^4 : x^1 = x^2 = x^4 = 0\}, \quad L_0 = \{x \in \mathbb{R}^4 : x^4 = 0\}, \quad x = (x^\mu),$$

and

$$J_0[(x^1, x^2, 0, 0)] = [(-x^2, x^1, 0, 0)]$$

where square brackets denote an equivalence class mod K_0. The *optical group*

$$G_0 \subset GL(4, \mathbb{R})$$

is the group of all optical automorphisms of the standard structure; it is a 9-dimensional Lie group consisting of all matrices of the form

$$\begin{pmatrix} \rho\cos\phi & \rho\sin\phi & 0 & p \\ -\rho\sin\phi & \rho\cos\phi & 0 & q \\ a & b & \sigma & r \\ 0 & 0 & 0 & \tau \end{pmatrix} \tag{15}$$

where $0 \leqslant \phi < 2\pi$, $\rho, \sigma, \tau \neq 0$ and $a, b, p, q, r \in \mathbb{R}$. An *optical frame* is an optical isomorphism e of the standard structure onto (V, K, L, J). If e is optical and $a \in G_0$, then ea is optical and all optical frames can be so obtained from one of them. Let $e_\mu (\mu = 1, 2, 3, 4)$ be the image by e of the μth unit coordinate vector in $\mathbb{R}^4$, then $e_3 \in K$, the vectors e_1, e_2 and e_3 span L and $J(e_1 \bmod K) = e_2 \bmod K$. One usually identifies e with (e_μ).

Let Op $F(V) \subset F(V)$ be the set of all optical frames of (V, K, L, J) and consider the map

$$g : \mathrm{Op}\, F(V) \to S(V)$$

defined by

$$g(e) = e^3 \otimes e^4 + e^4 \otimes e^3 - e^1 \otimes e^1 - e^2 \otimes e^2,$$

where (e^μ) is the frame dual to $e = (e_\mu)$ so that

$$L = \ker e^4.$$

If $a \in G_0$ is given by (15), then

$$g(ea) = \rho^{\ 2} g(e) + \xi \otimes e^4 + e^4 \otimes \xi \quad \text{for some} \quad \xi \in V^*.$$

It is now clear that condition (B) in the definition of the optical structure can be replaced by the following:

(B″) an orientation in L/K and a subset E of $S(V)$ such that, if $0 \neq k \in K$, then for each $g \in E$ one has $L = \ker \varkappa$, where $\varkappa = g(k)$; if g and $g' \in E$ then there is a number $\rho \neq 0$ and a 1-form ξ such that

$$g' = \rho^{-2} g + \xi \otimes \varkappa + \varkappa \otimes \xi. \tag{16}$$

Let F be a 2-form in an oriented vector space V with optical structure and let $\varkappa = g(k)$ be as in (B″). The conditions

$$k \lrcorner F = 0 \quad \text{and} \quad \varkappa \wedge F = 0 \tag{17}$$

depend only on the optical structure of V and not on the particular choice of k and g. A 2-form F satisfying (17) is said to be optical. It is known (see, for example, [7] and the references listed there) that, if F is optical and $g \in E$, then the Hodge dual $\sigma(g)F$ is also optical. Moreover, there holds

PROPOSITION 1. *If F is optical and $g, g' \in E$, then*

$$\sigma(g')F = \sigma(g)F$$

i.e. the Hodge dual of optical 2-forms is invariant under the deformations of the scalar product given by (16).

• Let M be an oriented, 4-dimensional differential manifold. An *optical geometry* on M is defined in several equivalent ways:

(a) as a smooth distribution of optical structures in the tangent spaces to M;

(b) as a G_0-structure on M;

(c) as a 1-dimensional distribution on M (vector subbundle) $\mathscr{K} \subset TM$ together with a class $\mathscr{E}$ of metric tensors on M, of Lorentz signature, and such that:

if $k_x \in \mathscr{K}_x \subset T_x M$ and $g \in \mathscr{E}$, then $g_x(k_x, k_x) = 0$, $x \in M$;

if g and $g' \in \mathscr{E}$, then g_x and g'_x are related to each other as in (16), where $\varkappa = g_x(k_x)$.

Let g be a Lorentz metric tensor and k a vector field on M which is optical with respect to g and nowhere zero on M. The pair (g, k) defines an optical geometry on M. Another such pair (g', k') defines the same optical geometry if and only if, for any $x \in M$, the vectors k_x and k'_x are parallel and the scalar products g_x and g'_x are related to each other as in (16).

The general theory of G-structures [8] provides us with ready definitions of optical isomorphisms, automorphisms and with the notion of integrability of a G_0-structure.

The following propositions follows from [7]:

PROPOSITION 2. *The G_0-structure on M defined by the pair (g, k) is integrable if and only if the distribution $\mathscr{L} = \ker g(k) \subset TM$ is integrable and the optical congruence generated by k consists of shear-free geodesics.*

PROPOSITION 3. *The flow generated by a vector field k on M consists of automorphisms of the optical geometry defined by (g, k) if and only if the lines of the flow form a congruence of shear-free, optical geodesics.*

With any optical geometry on M there is associated a complex line bundle $\mathscr{L}/\mathscr{K} \to M$ such that the fibre $(\mathscr{L}/\mathscr{K})_x = \mathscr{L}_x/\mathscr{K}_x$ has complex structure defined by J_x.

The Lie algebra of G_0 is of infinite type and, therefore, the group of automorphisms of an optical geometry need not be a Lie transformation group [8]. Optical automorphisms may be used to obtain new solutions of Maxwell's equations form old ones. A full account of optical geometry and its relation to algebraically special solutions of Einstein's equations will be presented elsewhere [9, 10].

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